TOPIC 3: Limits and Continuity

A. LIMIT OF A FUNCTION

1. Definition of Limit

<u>Intuitive Definition:</u>
Let f be a function defined on an open interval (a, b) containing c, except possibly at c itself. If $f(x)$ gets arbitrarily close to a number L for all x sufficiently close to c (on either side of c) but not equal to c, then we say that f approaches the limit L as x approaches c, and we write

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c.$$

and say "the limit of $f(x)$, as x approaches c, equals L".
(Sometimes, we even say in a shorter form: the limit of f at c is L.)

<u>Example</u>: Find the limit of $3x^2 - 1$ as x approaches 0.

X	*f*(*x*)	*x*	*f*(*x*)
-0.1	-0.97	0.1	-0.97
-0.01	-0.9997	0.01	-0.9997
-0.001	-0.999997	0.001	-0.999997
-0.0001	-0.99999997	0.0001	-0.99999997

$$\text{As } x \to 0,\ f(x) \to -1. \text{ So, } \lim_{x \to 0}\left(3x^2 - 1\right) = -1$$

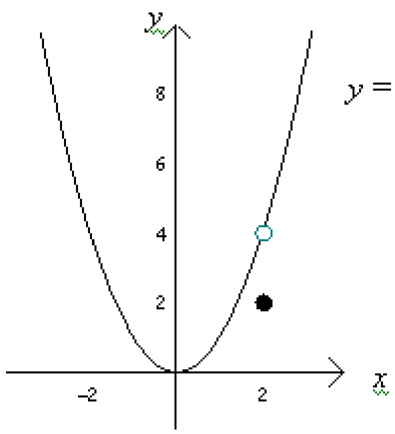
If no such number L exists, we say that f has no limit at c (i.e. $\lim_{x \to c} f(x)$ does not exist).

Notice that the limit does not depend on how the function is defined at c. The limit may exist even if the value of f at c is not known or undefined.

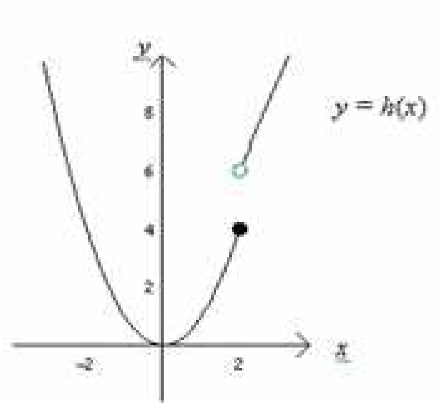
<u>Example</u>:

Find the limit of $g(x) = \begin{cases} x^2, x \neq 2 \\ 2, x = 2 \end{cases}$ and the limit of $h(x) = \begin{cases} x^2, x \leq 2 \\ 3x, x > 2 \end{cases}$, as x approaches 2.

<u>Solution:</u>

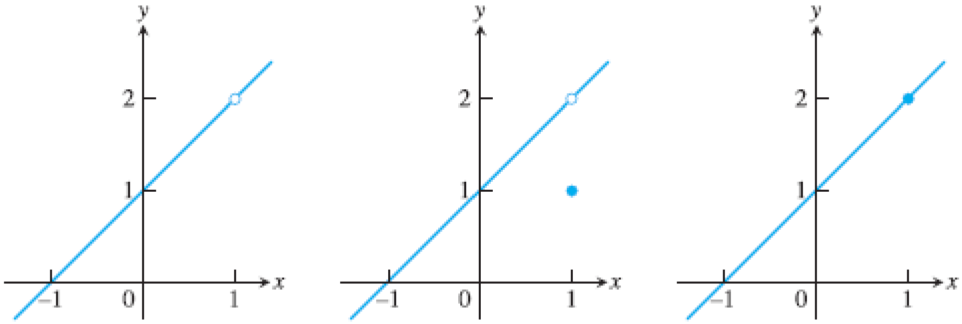


$\lim_{x \to 2} g(x) = 4$



$\lim_{x \to 2} h(x)$ ***does not exist***

Example:



(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

(T*)

$\lim\limits_{x\to 1} f(x) = 2$, $\lim\limits_{x\to 1} g(x) = 2$, $\lim\limits_{x\to 1} h(x) = 2$

Definition:

More formally, we say that the limit of $f(x)$ as x approaches c is L if for every number $\varepsilon > 0$ there is a corresponding number $\delta = \delta_\varepsilon > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta$$

[***For our course, this formal definition will not be used***.]

2. Limit Laws

Suppose $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = M$.		
1.	Uniqueness:	$\lim\limits_{x\to c} f(x) = K$ implies $K = L$, i.e. a function has at most one limit at a particular number
2.	Sum Rule:	$\lim\limits_{x\to c}[f(x) + g(x)] = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = L + M$
3.	Difference Rule:	$\lim\limits_{x\to c}[f(x) - g(x)] = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x) = L - M$
4.	Product Rule:	$\lim\limits_{x\to c}[f(x)g(x)] = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x) = L \cdot M$
5.	Constant Multiple Rule:	$\lim\limits_{x\to c} kf(x) = k \cdot \lim\limits_{x\to c} f(x) = k \cdot L$ for any $k \in R$
6.	Quotient Rule:	$\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \dfrac{L}{M}$ provided $M \neq 0$
7.	Power Rule:	$\lim\limits_{x\to c}[f(x)]^n = L^n$, n a positive integer
8.	Root Rule:	$\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, n a positive integer [If n is even, we assume that $\lim\limits_{x\to c} f(x) = L > 0$]

(*Can you state the above rules verbally*?)

Some easy and useful limits:

a) $\lim\limits_{x\to c} a = a$

b) $\lim\limits_{x\to c} x = c$

c) $\lim\limits_{x\to c} x^n = c^n$, where n is a positive integer

d) $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$, where n is a positive integer

(and if n is even, we assume that $c > 0$)

We shall try to use the above rules and easy limits in the following examples.

Example:

Evaluate the following limits, if they exist.

a) $\lim\limits_{x\to 2}\left(x^2 - 4x + 1\right)$ b) $\lim\limits_{x\to 3}\dfrac{x-2}{x+2}$ c) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$

d) $\lim\limits_{x\to 3}\dfrac{x-2}{x^2-4}$ e) $\lim\limits_{x\to -2}\dfrac{x^3+2x^2-1}{5-3x}$ f) $\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}$

g) $\lim\limits_{x\to 1}\dfrac{2x+1}{4x^2-1}$ h) $\lim\limits_{x\to -2}\sqrt{4x^2-3}$ i) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$

j) $\lim\limits_{x\to 0}\dfrac{(4+x)^2-16}{x}$ k) $\lim\limits_{x\to 2}\sqrt{2x^2-3}$ l) $\lim\limits_{x\to 1}\left(x^2-2\right)^{1/3}$

Solution:

Warning: ***If the instruction requires you to show some steps, you must do so or else you would lose marks.***

a)
$$\lim_{x\to 2}\left(x^2-4x+1\right) = \lim_{x\to 2} x^2 - \lim_{x\to 2} 4x + \lim_{x\to 2} 1$$
$$= 2^2 - 4(2) + 1 = \ldots = -3$$

$[\lim\limits_{x\to 2} x^2 = \lim\limits_{x\to 2} x \cdot \lim\limits_{x\to 2} x = 2\cdot 2 = 4]$

b)
$$\lim_{x\to 3}(x-2) = \lim_{x\to 3} x - \lim_{x\to 3} 2 = 3-2 = 1$$
$$\lim_{x\to 3}(x+2) = \lim_{x\to 3} x + \lim_{x\to 3} 2 = 3+2 = 5 \neq 0$$
$$\lim_{x\to 3}\frac{x-2}{x+2} = \frac{\lim\limits_{x\to 3}(x-2)}{\lim\limits_{x\to 3}(x+2)} = \frac{1}{5}$$

Sometimes, when you feel confident that the quotient rule can be applied, you may write the steps as:

$$\lim_{x\to 3}\frac{x-2}{x+2} = \frac{\lim\limits_{x\to 3}(x-2)}{\lim\limits_{x\to 3}(x+2)} = \frac{\lim\limits_{x\to 3} x - \lim\limits_{x\to 3} 2}{\lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 2} = \frac{3-2}{3+2} = \frac{1}{5}$$

(Sometimes one skips even more steps.)

A shorter way : $\lim\limits_{x\to 3}\dfrac{x-2}{x+2} = \dfrac{\lim\limits_{x\to 3}(x-2)}{\lim\limits_{x\to 3}(x+2)} = \dfrac{1}{5}$ [This way shows only *one intermediate step*.]

The shortest way: $\lim_{x\to 3}\frac{x-2}{x+2}=\frac{1}{5}$ [This way does not show any step at all; only the final answer is shown.]

Compare c) and d).

c) $\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{2+2}=4$

d) $\lim_{x\to 3}\frac{x-2}{x^2-4}=\frac{\lim_{x\to 3}(x-2)}{\lim_{x\to 3}(x^2-4)}=\frac{3-2}{3^2-4}=\frac{1}{5}$

Compare with $\lim_{x\to 3}\frac{x-2}{x^2-4}=\lim_{x\to 3}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 3}\frac{1}{x+2}=\frac{1}{3+2}=\frac{1}{5}$

e)

f) $\lim_{x\to 1}\frac{x-1}{x^2-1}$ (Why can't the quotient rule be applied?)

$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{(x-1)}{(x+1)(x-1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{\lim_{x\to 1}(x+1)}=\frac{1}{2}$

g)

h) $\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{4(-2)^2-3}=\ldots=\sqrt{13}$

i) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$ (***A critical step used***)

$=\lim_{x\to 0}\frac{???}{x(\sqrt{x+1}+1)}$

$=\lim_{x\to 0}\frac{???}{???}=$

j) $\lim_{x\to 0}\frac{(4+x)^2-16}{x}=\lim_{x\to 0}\frac{16+8x+x^2-16}{x}$

$=\lim_{x\to 0}\frac{??}{??}$

$=\lim_{x\to 0}(\qquad)=$

k)

l)

Direct Substitution Property

Limits of Polynomials

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial, then

$\lim_{x \to c} p(x) = p(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$.

Limits of Rational Functions

If $p(x)$ and $q(x)$ are polynomials and $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)}$$

Example: $p(x) = 4x^3 - 5x^2 + 3x - 4$

$\lim_{x \to 2}\left(4x^3 - 5x^2 + 3x - 4\right) = 4(2)^3 - 5(2)^2 + 3(2) - 4 = 14$, which is $p(2)$.

$\lim_{x \to 2} p(x) = p(2)$

Examples: 'Good case'

$$\lim_{x \to 2} \frac{4x^3 - 5x^2 + 3x - 4}{2x - 1} = ???.$$

$$\lim_{x \to 2}(2x - 1) = 3 \neq 0$$

$$\lim_{x \to 2} \frac{4x^3 - 5x^2 + 3x - 4}{2x - 1} = \frac{\lim_{x \to 2}(4x^3 - 5x^2 + 3x - 4)}{\lim_{x \to 2}(2x - 1)} = \frac{4(2)^3 - 5(2)^2 + 3(2) - 4}{2(2) - 1} = \frac{14}{3}$$

'Bad cases' (i) $\lim_{x \to 2} \dfrac{4x^3 - 5x^2 + 3x - 4}{2x - 4} = ???$

$$\lim_{x \to 2}(2x - 4) = 0$$

(ii) $\lim_{x \to 2} \dfrac{x^2 - 4}{2x - 4} = ???$

Another useful limit

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ (see note 1)

Reminder:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

[1] The derivation of this limit can be found in Stewart's Calculus, Thomas' Calculus and also other textbooks.

Example:

Evaluate the following limits, if they exist.

a) $\lim\limits_{x\to 0}\dfrac{\sin x}{2x}$ b) $\lim\limits_{x\to 0}\dfrac{(x-2)\sin x}{3x}$

3. Sandwich Theorem (Also known as **Squeezing Theorem** or **Pinching Theorem**)

Sandwich Theorem

If $f(x)\le g(x)\le h(x)$ for all x in an interval containing a number a, except possibly at a, and $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} h(x)=L$, then

$$\lim_{x\to a} g(x)=L.$$

Example:

a) If $x-x^2\le g(x)\le 4-3x$ for all x, find $\lim\limits_{x\to 2} g(x)$.

b) Evaluate $\lim\limits_{x\to 0} x^2\sin\dfrac{1}{x}$.

Solution:

a) Since $\lim\limits_{x\to 2}(x-x^2)=-2$, $\lim\limits_{x\to 2}(4-3x)=-2$, and $x-x^2\le g(x)\le 4-3x$,

by the Sandwich theorem, $\lim\limits_{x\to 2} g(x)=-2$.

b) $-1\le\sin\dfrac{1}{x}\le 1$, for all x except x = 0. Hence

$$-x^2\le x^2\sin\frac{1}{x}\le x^2$$

Since $\lim\limits_{x\to 0}(-x^2)=0=\lim\limits_{x\to 0} x^2$, by the Sandwich theorem,

$$\lim_{x\to 0} x^2\sin\frac{1}{x}=0.$$

A more general example

For any function, $\lim\limits_{x\to c}|f(x)|=0$ implies $\lim\limits_{x\to c} f(x)=0$

Since $-|f(x)|\le f(x)\le |f(x)|$ and $\lim\limits_{x\to c}-|f(x)|=\lim\limits_{x\to c}|f(x)|=0$,

by the Sandwich theorem, $\lim\limits_{x\to c} f(x)=0$.

3. One-sided Limits

Let f be a function defined on an open interval (c,d). If $f(x)$ gets arbitrarily close to a number L as x approaches c from within (c,d), i.e. x approaches c from the right, then we say that f has a **right-hand limit** L at c, and we write

$$\lim_{x\to c^+} f(x) = L \text{ or } f(x)\to L \text{ as } x\to c^+.$$

Note that how $f(x)$ is defined for $x \le c$ plays no role in this case.
"$x\to c^+$" means that we consider only values of *x* that are greater than *c*.

Similarly, if f is defined on an open interval (b,c) and gets arbitrarily close to a number M as x approaches c from within (b,c), i.e. x approaches c from the left, then we say that f has a **left-hand limit** M at c, and we write

$$\lim_{x\to c^-} f(x) = M \text{ or } f(x)\to M \text{ as } x\to c^-.$$

As in the previous case, how $f(x)$ is defined for $x \ge c$ plays no role in this case.
"$x\to c^-$" means that we consider only values of *x* that are greater than *c*.

Theorems:

a) The Limit Laws and The Sandwich Theorem are also valid for one-sided limits if $x\to c$ is replaced by $x\to c^-$ or $x\to c^+$ respectively

b) $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.
[*This would be very useful when dealing with piecewise-defined functions*,]

Example:
Determine if the limits exist.

(i) $f(x)=\begin{cases} x+2, x\le 0 \\ x-1, x>0\end{cases}$ a) $\lim_{x\to 0^-} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0} f(x)$

(ii) $f(x)=\begin{cases} 5x-1, x<4 \\ 4x+3, x\ge 4\end{cases}$ a) $\lim_{x\to 4^-} f(x)$ b) $\lim_{x\to 4^+} f(x)$ c) $\lim_{x\to 4} f(x)$

Solution:

詺閴扩퇃 $f(x)=\begin{cases} x+2, x\le 0 \\ x-1, x>0\end{cases}$

a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+2) = 2$ b) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-1) = -1$

c) Since $\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.

(ii) $f(x)=\begin{cases} 5x-1, x<4 \\ 4x+3, x\ge 4\end{cases}$ a) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5x-1) = 20-1 = 19$

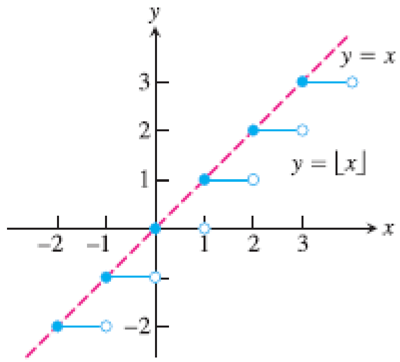
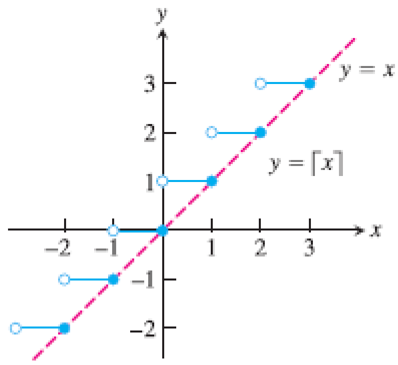
b) $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(4x+3) = 16+3 = 19$

c) Thus, $\lim_{x\to 4} f(x)$ exists because $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$.

$\lim_{x\to 4} f(x) = 19$

For a real number x, $\lfloor x \rfloor$ is the largest integer less than or equal to x. For example, $\lfloor 2 \rfloor = 2, \lfloor 2.5 \rfloor = 2, \lfloor -2.5 \rfloor = -3$. The function $f(x) = \lfloor x \rfloor$ is called the ***floor*** function.

For a real number x, $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, $\lceil 2 \rceil = 2, \lceil 2.5 \rceil = 3, \lceil -2.5 \rceil = -2$. The function $f(x) = \lceil x \rceil$ is called the ***ceiling*** function.

The floor function	The ceiling function
 (T*)	 (T*)
The graph of the least integer function $y = \lfloor x \rfloor$ lies on or above the line $y = x$, so it provides an integer floor for *x*.	The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for *x*.

<u>Example</u>:
Evaluate each of the following limits, if it exists. If it does not exist, explain why.

a) $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ b) $\lim\limits_{x \to 2} \lfloor x \rfloor$ c) $\lim\limits_{x \to 2} \lceil x \rceil$

<u>Solution</u>:

a) $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ $\qquad |x-2| = \begin{cases} -(x-2), & \text{if } x < 2 \\ x-2, & \text{if } x \geq 2 \end{cases}$

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)}{x-2} = \lim_{x \to 2^-} (-1) = -1$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = \lim_{x \to 2^+} 1 = 1$$

$\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ does not exist. (Why?)

b) $\lim\limits_{x \to 2} \lfloor x \rfloor$

For $x < 2$ and near 2, $\lfloor x \rfloor = 1$. So $\lim\limits_{x \to 2^-} \lfloor x \rfloor = \lim\limits_{x \to 2^-} 1 = 1$

For $x > 2$ and near 2, $\lfloor x \rfloor = 2$ So $\lim\limits_{x \to 2^+} \lfloor x \rfloor = \lim\limits_{x \to 2^-} 2 = 2$

c) $\lim\limits_{x \to 2} \lceil x \rceil$

For $x < 2$ and near 2, $\lceil x \rceil = 2$. So $\lim\limits_{x \to 2^-} \lceil x \rceil = \lim\limits_{x \to 2^-} 2 = 2$

For $x > 2$ and near 2, $\lceil x \rceil = ?$ So $\lim\limits_{x \to 2^+} \lceil x \rceil = ??$

B. CONTINUITY

1. Continuity Test

For a function f that is defined at least on an open interval about a number c, we say that f is **continuous at** c if and only if

1. $f(c)$ exists (i.e., *the value of* $f(c)$ *is defined*; *this condition is not necessary for the existence of limit*);
2. $\lim_{x \to c} f(x)$ exists; and
3. $\lim_{x \to c} f(x) = f(c)$.

[Summarized: "limit of f at c equals $f(c)$"]

If f is not continuous at c, we say that f is discontinuous at c. In this case, c is said to be a discontinuity of f.

When a function f is discontinuous at c, what sort of situation could occur?

<u>**Example**</u>:
Determine whether the following functions are continuous at $x = a$.

a) $f(x) = 4x^3 + 2x + 1$; $a = 0$

b) $f(x) = \dfrac{2x+3}{3x-2}$; $a = \frac{2}{3}$

c) $f(x) = \begin{cases} \dfrac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$; $a = 0$

d) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$; $a = 2$

e) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & if \quad x \neq 2; \\ 3 & if \quad x = 2. \end{cases}$; $a = 2$

<u>**Solution:**</u>

a) $f(x) = 4x^3 + 2x + 1$; $a = 0$

Since (i) $f(x)$ is defined at $x = 0$ with $f(0) = 1$,
(ii) $\lim_{x \to 0} f(x)$ exist with $\lim_{x \to 0} f(x) = 1$, and
(iii) $\lim_{x \to 0} f(x) = f(0)$,

$f(x) = 4x^3 + 2x + 1$ is continuous at $a = 0$.

b) $f(x) = \dfrac{2x+3}{3x-2}$; $a = \frac{2}{3}$ $\quad f(\frac{2}{3})$ undefined. ***Conclusion***?

c) $f(x) = \begin{cases} \dfrac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$; $a = 0$ $\quad f(0) = 1$ [f(0) is defined]

$\lim_{x \to 0} \dfrac{1}{x^2}$ does not exist. ***Conclusion***?

d) $f(x)=\dfrac{x^2-x-2}{x-2}$; $a=2$ $f(2)$ undefined. **Conclusion**?

This function is not the same as $g(x)=x+1$. Why???

e) $f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2} & if \quad x\neq 2;\\ 3 & if \quad x=2.\end{cases}$; $a=2$

[This function is the same as $g(x)=x+1$. Why???]

$$f(2)=3,\ \lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\ldots=\lim_{x\to 2}(x+1)=3$$

2. Continuity Rules

Theorem

If the functions f and g are continuous at a, then the following functions are continuous at a.

1	Sum:	$f+g$
2	Difference:	$f-g$
3	Product:	$f\cdot g$
4	Constant Multiple:	$c\cdot f$ for any $c\in R$
5	Quotient:	$\dfrac{f}{g}$ provided $g(a)\neq 0$

Theorems and Observations:

1. Any polynomial is continuous everywhere, i.e., it is continuous on $R=(-\infty,\infty)$.
2. The functions sin x and cos x are continuous at any number c.
3. The function tan x is continuous everywhere EXCEPT at $\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\cdots$
4. $f(x)=\dfrac{1}{x-c}$ is continuous everywhere except at the number c.

   Indeed, $\lim\limits_{x\to c} f(x)$ does not exist.
5. Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

6. The following types of functions are continuous at every number in their domains:

   polynomials rational functions

   root functions trigonometric functions

Examples: On what intervals is each function continuous?

$$f(x)=x^{2012}-12x^{57}+1900,\ g(x)=\frac{x+1}{x^2-2x},\ h(x)=\sqrt{x}+\frac{x}{x-2},\ m(x)=\frac{\cos x}{3+\sin x}$$

3. Composite of Continuous Functions

Theorem:
If f is continuous at a, and g is continuous at $f(a)$, then the composite $g \circ f$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

Example:
Determine whether the following functions are continuous.

a) $h(x) = \cos(x^2)$ b) $k(x) = \dfrac{1}{\sqrt{x^2+9}-5}$

Solution:
a) We have $h(x) = g(f(x))$, where

$$f(x) = x^2 \text{ and } g(x) = \cos x$$

Now f is continuous on R since it is a polynomial, and g is also continuous everywhere. Thus, $h = g \circ f$ is continuous on R by the above theorem.

b) Notice that k can be written as the composition of four functions:

$$k = r \circ h \circ g \circ f \quad \text{or} \quad k(x) = r(h(g(f(x))))$$

where $r(x) = \dfrac{1}{x}$, $h(x) = x - 5$, $g(x) = \sqrt{x}$, $f(x) = x^2 + 9$

We know each of these functions is continuous on its domain, so by the above theorem, k is continuous on its domain, which is

$\{x \in R \mid \sqrt{x^2+9} \neq 5\} = \{x \mid x \neq \pm 4\} = (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

Example:

Find the following limits if they exist. (Here, try to make use of continuity of a function.)

a) $\lim_{x \to 3} 5\cos(x^2 - 9)$ b) $\lim_{x \to \pi} 2\sin^2 x - 3$

4. Continuity on an interval

Before discussing the continuity of a function on an interval, we need to discuss one-sided continuity.

Definition: **Continuity from the left and right (One-sided continuity)**

A function f is **continuous from the left at the point *a*** if the following conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x \to a^-} f(x)$ exists.
3. $\lim_{x \to a^-} f(x) = f(a)$

Similar definition for
- f is **continuous from the right** at the point a

Definition: Continuity on an interval

- A function f is **continuous on the open interval** (a,b) if f is continuous at all points of the open interval (a,b) .
- A function f is **continuous on the closed interval** $[a,b]$ if f is continuous on the open interval (a,b), continuous from the right at a and continuous from the left at b.
- "f is continuous on $(-\infty,\infty)$" means "f is continuous everywhere".

Example

$$f(x)=\begin{cases} -x & if \;\; x<0 \\ x & if \;\; 0\le x\le 1 \\ x+1 & if \;\; x>1 \end{cases}$$

Find each of the following, or, if it does not exist, explain why.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 1} f(x)$ (c) $f(1)$ (d) $\lim_{x\to 1^+} f(x)$

Discuss continuity of f on intervals.

Example

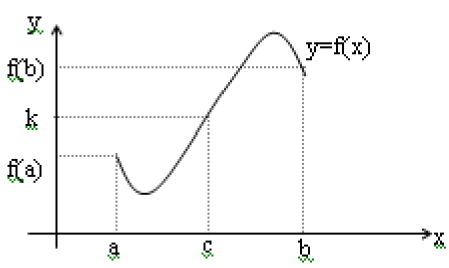
Where are each of the following functions discontinuous?

(a) $f(x)=\dfrac{x^2-x-2}{x-2}$ (b) $g(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & if \quad x\neq 2; \\ 2 & if \quad x=2. \end{cases}$

(c) $h(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & if \quad x\neq 2; \\ 3 & if \quad x=2. \end{cases}$

Discuss continuity of the functions on intervals.

5. Intermediate Value Theorem for Continuous Functions



Suppose f is a continuous function on a closed interval $[a,b]$. If k is a number such that $f(a)<k<f(b)$ or $f(b)<k<f(a)$, then there is a number $c\in(a,b)$ with $f(c)=k$.

[***Note***: This theorem does not tell us what c is.]

Example:

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.

f is continuous on the closed interval [1, 2]. [f is continuous since it is a polynomial.]

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Take k = 0 in the theorem.

Since $f(1) < 0 < f(2)$, [0 is a number between $f(1)$ and $f(2)$.]

By the Intermediate Value Theorem, there is a number c between 1 and 2 such that $f(c) = 0$.

Therefore, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval (1, 2).

C. LIMITS INVOLVING INFINITY

1. Limits at Infinity and Horizontal Asymptotes

Definition: Limits at Infinity

We say that $f(x)$ has the limit L as x approaches infinity (∞) and write

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

if, as x moves further and further away from the origin in the positive direction, $f(x)$ gets arbitrarily close to L.

Analogously, we say that $f(x)$ has the limit M as x approaches minus infinity (–∞) and write $\lim_{x\to-\infty} f(x) = M$ or $f(x) \to M$ as $x \to -\infty$

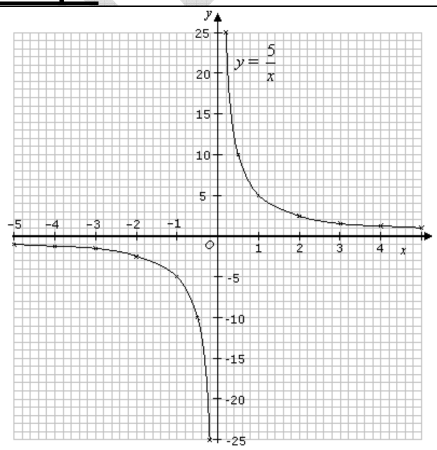
if, as x moves further and further away from the origin in the negative direction, $f(x)$ gets arbitrarily close to M.

Definition

A line $y = L$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example



$$\lim_{x\to\infty} \frac{5}{x} = 0 \qquad \lim_{x\to-\infty} \frac{5}{x} = 0$$

The line $y = 0$ is a horizontal asymptote of the curve $y = \frac{5}{x}$

What can you say about

$\lim_{x\to\infty} \frac{1}{x^2}$ and $\lim_{x\to-\infty} \frac{1}{x^2}$?

Limit Laws

Suppose $\lim\limits_{x\to c} f(x)=L$ and $\lim\limits_{x\to c} g(x)=M$, and $\lim\limits_{x\to c}$ means $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to-\infty}$.

1. Uniqueness: $\lim\limits_{x\to c} f(x)=K$ implies $K=L$, i.e. a function has at most one limit as $x\to\infty$ (or as $x\to-\infty$).
2. Sum Rule: $\lim\limits_{x\to c}\left[f(x)+g(x)\right]=\lim\limits_{x\to c} f(x)+\lim\limits_{x\to c} g(x)=L+M$
3. Difference Rule: $\lim\limits_{x\to c}\left[f(x)-g(x)\right]=\lim\limits_{x\to c} f(x)-\lim\limits_{x\to c} g(x)=L-M$
4. Product Rule: $\lim\limits_{x\to c} f(x)g(x)=\lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)=L\cdot M$
5. Constant Multiple Rule: $\lim\limits_{x\to c} kf(x)=k\cdot\lim\limits_{x\to c} f(x)=k\cdot L$ for any $k\in R$
6. Quotient Rule: $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}=\dfrac{L}{M}$ provided $M\neq 0$
7. Power Rule: $\lim\limits_{x\to c}[f(x)]^n=L^n$, n a positive integer
8. Root Rule: $\lim\limits_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{L}=L^{\frac{1}{n}}$, n a positive integer

[If n is even, we assume that $\lim\limits_{x\to c} f(x)=L>0$]

Example

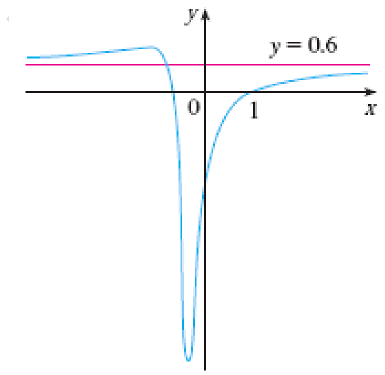
(a)

When x becomes large, both the numerator and the denominator of $\dfrac{3x^2-x-2}{5x^2+4x+1}$ become large, so it is not obvious what happens to the ratio.

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$=\frac{\lim\limits_{x\to\infty}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)}=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty}5+\lim\limits_{x\to\infty}\frac{4}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}=\frac{3-0-0}{5+0+0}=\frac{3}{5}$$

Show steps to arrive at $\lim\limits_{x\to-\infty}\dfrac{3x^2-x-2}{5x^2+4x+1}=\dfrac{3}{5}$.



$y=\dfrac{3}{5}$ is a horizontal asymptote of the curve $y=\dfrac{3x^2-x-2}{5x^2+4x+1}$.

(b)

$$\lim_{x\to-\infty}\frac{3x+2}{5x^3-4}=\lim_{x\to-\infty}\frac{\frac{3x+2}{x^3}}{\frac{5x^3-4}{x^3}}=\lim_{x\to-\infty}\frac{\frac{3}{x^2}+\frac{2}{x^3}}{5-\frac{4}{x^3}}=\frac{\lim\limits_{x\to-\infty}\left(\frac{3}{x^2}+\frac{2}{x^3}\right)}{\lim\limits_{x\to-\infty}\left(5-\frac{4}{x^3}\right)}=\frac{\lim\limits_{x\to-\infty}\frac{3}{x^2}+\lim\limits_{x\to-\infty}\frac{2}{x^3}}{\lim\limits_{x\to-\infty}5-\lim\limits_{x\to-\infty}\frac{4}{x^3}}=\frac{0+0}{5-0}=0$$

(c) $\lim\limits_{x\to\infty}\dfrac{2x^2+5}{3x+1}$

[**Note**: In (a) the numerator and the denominator of the rational function have the same degree; in (b) the degree of the numerator is less than the degree of the denominator.
In example (c), the degree of the numerator is greater than the degree of the denominator; it will be discussed in the next subsection under infinite limits.]

Example

Use the rules for limits at infinity to evaluate the following limits.

a) $\lim_{x\to\infty}\frac{3x+2}{5x-4}$ b) $\lim_{x\to\infty}\frac{2x^2+8x+6}{x^2-3x+1}$

Solution:

a)
$$\lim_{x\to\infty}\frac{3x+2}{5x-4}=\lim_{x\to\infty}\frac{3+\frac{2}{x}}{5-\frac{4}{x}}$$
$$=\frac{\lim_{x\to\infty}\left(3+\frac{2}{x}\right)}{\lim_{x\to\infty}\left(5-\frac{4}{x}\right)}=\frac{\lim_{x\to\infty}3+\lim_{x\to\infty}\frac{2}{x}}{\lim_{x\to\infty}5-\lim_{x\to\infty}\frac{4}{x}}$$
$$=\frac{3+0}{5-0}=\frac{3}{5}$$

2. Infinite Limits and Vertical Asymptotes

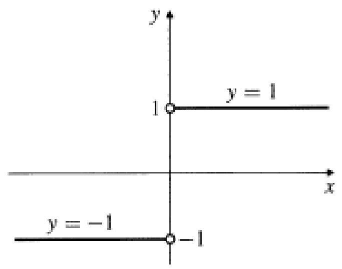
Example (a) Let's try to decide if $\lim_{x\to 0}\frac{1}{x^2}$ exists.

As x approaches 0, x^2 also becomes close to 0 and $\frac{1}{x^2}$ becomes very large; the values of $f(x)=\frac{1}{x^2}$ do not approach a number. We conclude that $\lim_{x\to 0}\frac{1}{x^2}$ ***does not exist***.

However in this example, the values of $f(x)=\frac{1}{x^2}$ can be made arbitrarily large by taking x close enough to 0.

We write $\lim_{x\to 0}\frac{1}{x^2}=\infty$ in addition to the information that "$\lim_{x\to 0}\frac{1}{x^2}$ ***does not exist***".

Example (b) Consider $s(x)=\frac{x}{|x|}=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\\ \text{undefined} & \text{if } x=0\end{cases}$

	$\lim_{x\to 0^+}\frac{x}{\lvert x\rvert}=\lim_{x\to 0^+}1=1$ $\lim_{x\to 0^-}\frac{x}{\lvert x\rvert}=\lim_{x\to 0^-}-1=-1$ $\lim_{x\to 0}\frac{x}{\lvert x\rvert}$ ***does not exist.*** But $\cancel{\lim_{x\to 0}\frac{x}{\lvert x\rvert}}=\infty$ [We cannot write this way.]

Definition of infinite limits

We say that $f(x)$ approaches infinity as x approaches c, and we write

$$\lim_{x\to c} f(x) = \infty$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x-c| < \delta \Rightarrow f(x) > B$$

Analogously, we say that $f(x)$ approaches minus infinity as x approaches c, and we write

$$\lim_{x\to c} f(x) = -\infty$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x-c| < \delta \Rightarrow f(x) < -B$$

One-sided infinite limits like $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^+} f(x) = -\infty$, $\lim_{x\to c^-} f(x) = \infty$ and $\lim_{x\to c^-} f(x) = -\infty$, are similarly defined by confining values of x to one side of c.

Infinite limits at infinity

There are also situations where $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = -\infty$,

Definition

A line $x = c$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if

either $\lim_{x\to c^+} f(x) = \infty \text{ or } -\infty$ or $\lim_{x\to c^-} f(x) = \infty \text{ or } -\infty$

<u>Remark</u>: ∞ and $-\infty$ are not real numbers; they are symbols. Writing $\lim_{x\to c} f(x) = \infty$ or $\lim_{x\to c} f(x) = -\infty$ does not mean that the limit exists, although these are given the names infinite limits.

<u>Example</u>

$$\lim_{x\to\infty} \frac{2x^2+5}{3x+1} = \lim_{x\to\infty} \frac{(2x^2+5)/x}{(3x+1)/x} = \lim_{x\to\infty} \frac{\frac{2x^2+5}{x}}{\frac{3x+1}{x}} = \lim_{x\to\infty} \frac{2x+\frac{5}{x}}{3+\frac{1}{x}} = \infty$$

What about $\lim_{x\to-\infty} \frac{2x^2+5}{3x+1}$?

Example:

The following limits do not exist (as real numbers). Write each limit as ∞ or $-\infty$.

a) $\lim\limits_{x\to 3^+} \dfrac{-6}{x-3}$ b) $\lim\limits_{x\to 1} \dfrac{2}{(x-1)^2}$ c) $\lim\limits_{x\to 2^-} \dfrac{-3}{x-2}$

d) $\lim\limits_{x\to -\infty} \dfrac{x^2-3}{2x-4}$ e) $\lim\limits_{x\to 0} \dfrac{-1}{x^2(x+1)}$ f)

Solution:

a) $\lim\limits_{x\to 3^+} \dfrac{-6}{x-3}$

Since for $x > 3$, $(x-3) > 0$ and $\lim\limits_{x\to 3^+}(x-3) = 0$ thus

$$\lim_{x\to 3^+} \frac{-6}{x-3} = -\infty$$

3. Horizontal and Vertical Asymptotes

Finding horizontal and vertical asymptotes of the graph of a rational function is quite easy.

Example:

(i). Determine the horizontal asymptote(s) for the graph of each function defined below.

a) $f(x) = \dfrac{2x+1}{x-4}$ b) $f(x) = \dfrac{8x^2-1}{1+4x+6x^2}$

(ii) Determine the vertical asymptote(s) for the graph of each function defined below.

a) $f(x) = \dfrac{-3}{x+2}$ b) $f(x) = \dfrac{2}{1-x}$ c) $f(x) = \dfrac{1}{x^2-5x+4}$

Solution:

(i) a) $f(x) = \dfrac{2x+1}{x-4}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+1}{x-4} = \ldots = 2$$

Thus the horizontal asymptote is $y = 2$.

(ii) For **vertical asymptote**: consider $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$

a) $f(x) = \dfrac{-3}{x+2}$

$\lim\limits_{x\to -2^-} \dfrac{-3}{x+2} = \infty$ or $-\infty$??? $\lim\limits_{x\to -2^+} \dfrac{-3}{x+2} = \infty$ or $-\infty$???

Since $f(x) \to \infty$ as $x \to -2^-$ [or $f(x) \to -\infty$ as $x \to -2^+$],
the vertical asymptote is $x = -2$.

(nby, Nov 2015)